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Conservation of extremal ellipticity for coherent single mode Gaussian beams propagating in rotationally invariant media

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ABSTRACT

We demonstrate that the conservation of extremal ellipticity for an astigmatic Gaussian beam propagating in a rotationally invariant medium, for example, an arbitrary sequence of free space propagation and stigmatic lenses, initially identified by Lo et al. (2017), is a direct consequence of the conservation of orbital angular momentum (OAM) of the beam. We derive explicit analytical expressions for the first few non-zero moments of the beam OAM and show that, apart from fundamental constants, they depend only on the beam's extremal ellipticity. This unexpected connection uncovers the actual origin of a new propagation invariant, namely extremal ellipticity, for a broad class of coherent, single-mode Gaussian beams. We expect this principle to find use in various applications where spot circularity along an extended region of the beam is essential; such as laser micromachining, imaging, optical trapping, and inertial confinement fusion.

1. Introduction

It is now well known that the orbital angular momentum (OAM) of light depends on the spatial electric field amplitude and phase structure of the beam wavefront. Since the original proposal by Allen et al. in 1992 [1], OAM has turned out to be an interesting and promising area of study [2], with applications such as angular momentum transfer [3], rotational Doppler effect [4,5], optical manipulation of microscopic particles [6] and cold atoms [7], phase contrast microscopy [8], stimulated emission depletion microscopy [9], quantum [10] as well as classical communication [11], superresolution microscopy [12], and optical tweezers [13,14].

Many beams carrying OAM possess a phase singularity and a helical wavefront. However, Padgett [15] has shown that certain Gaussian beams focused by a cylindrical lens can possess large OAM without a phase singularity. Here, specifically, we address vortex-free general astigmatic Gaussian beams [16,17], which can carry non-zero OAM. We limit ourselves to fully coherent, single-mode beams that can be described in an electric field representation. The treatment of more general beams like partially coherent Gauss–Schell model beams [17–19], vortex Hermite–Gaussian beams [20], astigmatic Hermite–Gaussian beams with vortex-induced OAM [21], Laguerre-Gaussian beams integrally transformed from Hermite–Gaussian beams under astigmatism influence [22] or incoherent superpositions of several transverse modes, e.g., Siegman's "non-Gaussian Gaussian" [23] may be considered in the future, but is beyond the scope of this work.

This work investigates the physical origin underlying the effect discovered by Lo et al. [24], which demonstrated that the extremal ellipticity for simple as well as general astigmatic Gaussian beams is a conserved quantity upon propagation through free space and stigmatic lenses and is solely dependent on the input beam parameters astigmatism A and asymmetry Ω, defined below. This finding is of great importance for the specifications of industrial lasers and the design of laser micromachining systems [25,26]. Here, we further elucidate that the conservation of extremal ellipticity [24] upon transmission through a rotationally invariant medium is a fundamental and direct consequence of the conservation of the OAM of the Gaussian beam. We extensively show that this is applicable not just under very limited circumstances, but is a robust phenomenon across rather general classes of optical beams. Furthermore, the extremal ellipticity gives us a tool to manipulate the OAM distribution of a broad class of astigmatic Gaussian beams.

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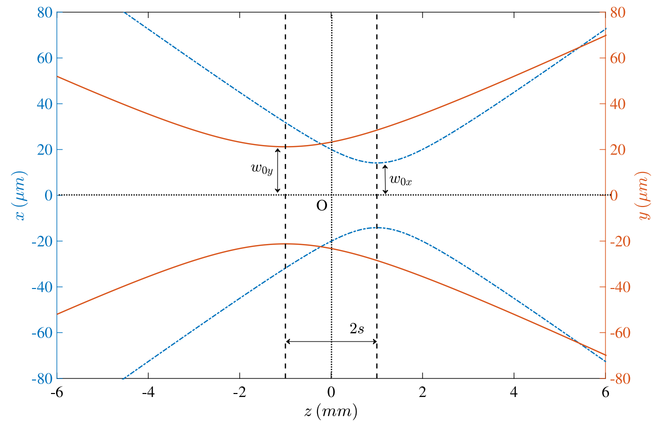

Fig. 1. The intensity profiles of the x-z (blue dot-dashed) and y-z (orange solid) cross-sections for the fundamental mode of a simple astigmatic Gaussian beam propagating along z-direction. 'O' is the origin of the coordinate system in use. The distance between the waists is $2s = 2$ mm, the spot half-widths along x and y directions are $w_{0x} = 10\sqrt{2}\,\mu\text{m} = 14.14\,\mu\text{m}$ and $w_{0y} = 15\sqrt{2}\,\mu\text{m} = 21.21\,\mu\text{m}$, respectively, and the wavelength (λ) is 632.8 nm. Thus, the Rayleigh ranges are $z_{R_x} = 0.993$ mm and $z_{R_y} = 2.245$ mm, the astigmatism $A = 1.235$ and the asymmetry $\Omega = -0.385$.

In Section 2, we provide the background for studying simple astigmatic paraxial Gaussian beams. Section 3 incorporates the OAM calculation for the simple astigmatic case. Section 4 deals with higher-order Hermite–Gaussian modes for simple astigmatic Gaussian beams. In Section 5, we briefly describe some properties of general astigmatic beams and show the analytical form of the first moment of OAM. Section 6 investigates the first moment of OAM for higher-order Hermite–Gaussian modes of general astigmatic Gaussian beams. Conclusions are supplied in Section 7. Further details on the common definition of astigmatic Gaussian beams as well as calculations of higher order OAM moments are provided in the appendix.

2. Simple astigmatic coherent Gaussian beam: fundamental mode

Following Siegman [27], we consider an electromagnetic field $E(x, y, z)$ propagating in free space according to the Helmholtz equation, $\left(\nabla^2 + k^2\right) E(x, y, z) = 0$, which in the paraxial approximation (z-axis being the direction of propagation) reduces to

$$\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial y^2}u - 2ik\frac{\partial}{\partial z}u = 0\ , \tag{1}$$

where we have used $E(x, y, z) = u(x, y, z)e^{-ikz}$, with $u(x, y, z)$ being the mode function describing the transverse beam profile, and k is the wavenumber ($k = 2\pi/\lambda$). Here, we consider Gaussian beams, which are exact solutions of the paraxial Helmholtz equation [Eq. (1)].

2.1. Mode function

The mode function of a simple astigmatic Gaussian beam in the fundamental mode is generally written as [27]

$$u(x, y, z) = \sqrt{\frac{2}{\pi w_x w_y}} \exp\left[-i\,\frac{k}{2}\left(\frac{x^2}{q_x} + \frac{y^2}{q_y}\right) + i\,\left(\frac{\psi_{0x} + \psi_{0y}}{2}\right)\right]\ , \tag{2}$$

where [28,29]

$$q_x = z - s + iz_{R_x} \qquad \text{and} \qquad q_y = z + s + iz_{R_y}\ , \tag{3}$$

are the complex beam parameters with $z_{R_x} = kw_{0x}^2/2$ and $z_{R_y} = kw_{0y}^2/2$ being the Rayleigh ranges, s being the half-distance between the x- and y- beam waists and w_{0x} and w_{0y} being the beam waist half-widths (see Fig. 1). Furthermore, w_x and w_y are the spot half-widths (see Fig. 2) defining the boundary at which the irradiance drops to $1/e^2$ of the peak irradiance at any given propagation distance and ψ_{0x}, ψ_{0y} are the Gouy phases. The complex beam parameters can be decomposed into their real and imaginary components as $q_{x,y}^{-1} = R_{x,y}^{-1} - i\,(kw_{x,y}^2/2)^{-1}$, where $R_{x,y}$ are the wavefront radii of curvature, turning the above mode function into

$$\begin{aligned} u(x, y, z) = \sqrt{\frac{2}{\pi w_x w_y}} \exp\Bigg[& -\left(\frac{x^2}{w_x^2} + \frac{y^2}{w_y^2}\right) \\ & - i\,\frac{k}{2}\left(\frac{x^2}{R_x} + \frac{y^2}{R_y}\right) + i\,\left(\frac{\psi_{0x} + \psi_{0y}}{2}\right)\Bigg]\ . \end{aligned} \tag{4}$$

The explicit forms of beam parameters are detailed in Eq. (A.1). In an ideal – also called stigmatic – Gaussian beam, the two beam parameters and Gouy phases are identical, i.e., $q_x = q_y$, $w_x = w_y$, $R_x = R_y$, and $\psi_{0x} = \psi_{0y}$, making the beam possess a circularly symmetric Gaussian spot irradiance distribution and a spherical wavefront. When $q_x \neq q_y$, the beam becomes an astigmatic Gaussian beam with elliptical Gaussian spot irradiance distribution and ellipsoidal (or hyperboloidal) wavefront. Astigmatism is divided into two classes: (i) simple astigmatism (Fig. 2), where the axes of the ellipses of constant intensity and constant phase are aligned, and (ii) general astigmatism (Fig. 6), where they are no longer aligned.

To stay consistent with [24], we further define the relative astigmatic beam waist separation A between the locations of the waists in the x and y directions [Eq. (A.2)] (referred to as Astigmatism going forward) and the Rayleigh range asymmetry Ω (referred to simply as Asymmetry), defined by the normalized difference in the spot half-widths squared at the waist [Eq. (A.3)]. Also note that z_{R_x}, z_{R_y}, and s are the three basic, z-independent parameters of an astigmatic, aberration free, single mode Gaussian beam. In this paper, the calculational results will typically be presented either in terms of these three parameters or their normalized counterparts A and Ω.

2.2. Ellipticity: simple astigmatic beam

The ellipticity $\epsilon_0(z)$, which ranges from $-1 < \epsilon_0(z) < +1$, is defined as [24]

$$\epsilon_0(z) = \frac{w_x^2(z) - w_y^2(z)}{w_x^2(z) + w_y^2(z)}\ . \tag{5}$$

The "0" subscript is used to differentiate the ellipticity of this simple astigmatic beam from that of a general astigmatic beam (used in other sections). Note that $w_x(z)$ and $w_y(z)$ are the z-dependent spot half-widths expressed in Eq. (A.1), rather than the spot half-widths at the waists as in Eq. (A.3). Thus, the ellipticity varies with z as one might expect. By differentiating the ellipticity with respect to z, the extremal value of the ellipticity was found by Lo et al. [24] to be

$$\epsilon_{0_{\text{ext}}}^2 = \frac{A^2 + 4\Omega^2}{A^2 + 4} = \frac{4s^2 + (z_{R_x} - z_{R_y})^2}{4s^2 + (z_{R_x} + z_{R_y})^2}\ , \tag{6}$$

which occurs at

$$z_{\text{ext}} = \left(\frac{z_{R_x} + z_{R_y}}{2}\right)\frac{4\Omega \pm \sqrt{(4 + A^2)(A^2 + 4\Omega^2)}}{2A}. \tag{7}$$

This property is important because it has been proven by Lo et al. [24] that when a simple astigmatic Gaussian beam propagates through a stigmatic lens (with equal x- and y-optical power), the extremal ellipticity is conserved, i.e., *despite having different astigmatism and asymmetry from the incident beam, the beam after the lens has the same extremal ellipticity as the incident beam.* This has very practical implications for the specifications of industrial lasers and the design of laser micro-machining systems, where a certain minimal beam roundness is often required to ensure robust process windows [25,26]. Our paper provides an explanation for this phenomenon by relating the constancy of the extremal ellipticity to the conservation of OAM of the beam.

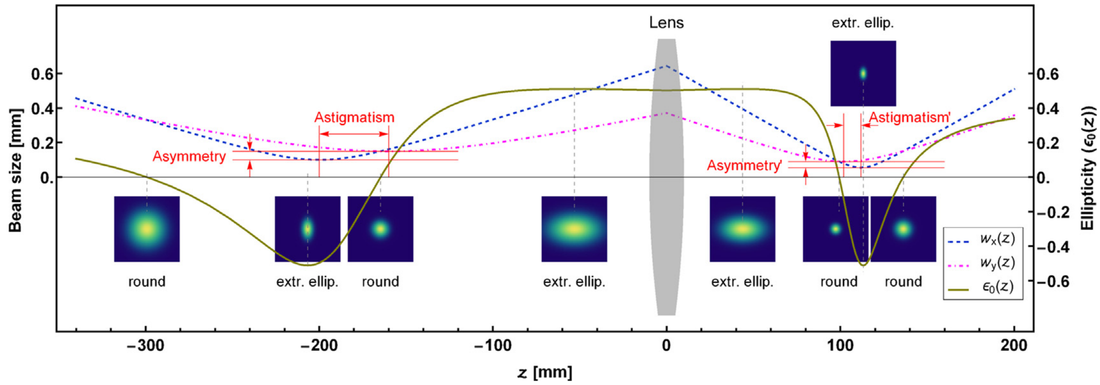

Fig. 2. The modulation of the intensity profile for a simple astigmatic Gaussian beam with $\lambda = 1\,\mu$m, $w_{0x} = 0.1$ mm, $w_{0y} = 0.15$ mm, x- and y- beam waists at propagation distances -200 mm and -160 mm, respectively (before the lens), propagating through a symmetric lens with focal length $f = 250$ mm placed at the origin ($z = 0$). The variation of beam widths ($w_{x(y)}$) both in the x- (dashed blue) and y- (dash-dotted magenta) directions and the resultant ellipticity $\epsilon_0(z)$ (solid olive) are shown along the propagation axes. Following the extremal ellipticity curve, we show corresponding beam cross-sections at different propagation distances, namely at positions of extremal ellipticity (extr. ellip.) and zero ellipticity (where the beam spot is round). The astigmatism (beam waists difference) and asymmetry (distance between beam waists) before and after (apostrophed) the lens have also been shown.

3. OAM-moments: simple astigmatic Gaussian beam in the fundamental mode

It is well-known that light has angular momentum, which can be expressed, in the case of paraxial beams, as the sum of spin angular momentum due to its polarization and OAM due to its spatial electric field amplitude and phase profile [1,30]. The expectation value of the OAM per photon is found by using the quantum mechanical angular momentum operator $L_z = -i\hbar(x\partial/\partial y - y\partial/\partial x)$ and the mode function $u(x, y, z)$ [1,31,32]

$$\langle L_z\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*(x, y, z) L_z u(x, y, z)\, dx\, dy\,, \tag{8}$$

where $u(x, y, z)$ is given by Eq. (4). The unit of $\langle L_z\rangle$ is the unit of $\hbar$, i.e. $\text{kg}\cdot\text{m}^2/\text{s}$. Eq. (8) evaluates to $\langle L_z\rangle = 0$ for all $u(x, y, z)$ given by Eq. (4) because the integrand is an odd function in both x and y.

It is important to note here that $\langle L_z\rangle = 0$ cannot be interpreted in our quantum formalism to imply that 'the beam has no angular momentum' or that the 'angular momentum of the beam is zero'. In this formalism, a beam can only be said to have no angular momentum when it is an eigenstate of angular momentum with eigenvalue zero. However, it can be readily established that the beam under consideration is *not* an eigenstate of angular momentum (i.e. applying $L_z u$ does not yield an eigenvalue times u); it is in fact a linear combination of an infinite number of angular momentum eigenstates. Therefore, the outcomes of possible angular momentum measurements follow a distribution; and only the *average* of that distribution is zero (i.e. $\langle L_z\rangle = 0$). As we will see next, other moments of the distribution need not vanish. However, they are constant, since the beam propagates through a rotationally invariant medium, and the whole angular momentum distribution, and therefore each moment, remains conserved.

The expectation value for the second moment of the z-orbital angular momentum per photon is given by

$$\langle L_z^2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*(x, y, z) L_z^2 u(x, y, z)\, dx\, dy\,, \tag{9}$$

and, with the mode function [Eq. (4)], evaluates to

$$\langle L_z^2\rangle = \hbar^2\left[\frac{w_x^2 w_y^2 k^2}{16}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)^2 + \frac{1}{4}\left(\frac{w_x}{w_y} - \frac{w_y}{w_x}\right)^2\right]. \tag{10}$$

The first contribution to $\langle L_z^2\rangle$ is due to the difference in wavefront curvatures, while the second one is due to the difference in the spot half-widths. Both terms vary with z, as shown in Fig. 3. However, their sum must necessarily be constant, as can indeed be seen by using Eqs. (A.1)–(A.3) in Eq. (10), which gives

$$\langle L_z^2\rangle = \hbar^2\left[\frac{s^2}{z_{R_x} z_{R_y}} + \frac{(z_{R_x} - z_{R_y})^2}{4 z_{R_x} z_{R_y}}\right] = \hbar^2\frac{A^2 + 4\Omega^2}{4(1-\Omega^2)}\,. \tag{11}$$

For an ideal Gaussian beam ($s = 0, z_{R_x} = z_{R_y}$), we note that Eq. (11) reduces to $\langle L_z^2\rangle = 0$. Importantly, Eq. (11) can be shown to be related to the extremal ellipticity $\epsilon_{0_{\text{ext}}}$ defined in Eq. (6), i.e.

$$\langle L_z^2\rangle = \hbar^2\left(\frac{\epsilon_{0_{\text{ext}}}^2}{1 - \epsilon_{0_{\text{ext}}}^2}\right), \tag{12}$$

from which we can see that, apart from the fundamental constant $\hbar$, the second moment of OAM is fully determined by the extremal ellipticity of the beam. Since $\langle L_z^2\rangle$ is conserved upon propagation through a rotationally symmetric optical systems, so is $\epsilon_{0_{\text{ext}}}$. We emphasize that this connection is neither physically nor mathematically obvious, and that therefore Eq. (12) is a nontrivial result. Higher moments of OAM, also sole functions of the extremal ellipticity, are presented in Appendix B.1.

4. Simple astigmatic coherent Gaussian beam: higher order modes

In this section, we will consider higher-order Hermite–Gaussian modes of a simple astigmatic beam, whose mode function is given by [27–29,31]

$$u_{mn}(x, y, z) = \left(\frac{1}{\pi w_x w_y 2^{m+n-1} m! n!}\right)^{1/2} H_m\left(\frac{\sqrt{2}x}{w_x}\right) H_n\left(\frac{\sqrt{2}y}{w_y}\right) \times \exp\left[-i\frac{k}{2}\left(\frac{x^2}{q_x} + \frac{y^2}{q_y}\right) + \frac{i}{2}\left((2m+1)\psi_{0x} + (2n+1)\psi_{0y}\right)\right]. \tag{13}$$

These modes are known to have $\langle L_z\rangle = 0$ just like the fundamental mode [1]. However, similar to the fundamental mode, the expectation value for the square of the OAM can be non-zero for higher order Hermite–Gaussian modes. Applying the same method as in Eq. (9) to find $\langle L_{z_{mn}}^2\rangle$ for a given mode $u_{mn}(x, y, z)$ returns an expression in terms of $\langle L_{z_{00}}^2\rangle$ of the fundamental mode described in Eq. (10),

$$\langle L_{z_{mn}}^2\rangle = (2m+1)(2n+1)\left(\langle L_{z_{00}}^2\rangle + \frac{\hbar^2}{2}\right) - \frac{\hbar^2}{2}, \tag{14}$$

where it can be confirmed that $\langle L_{z_{00}}^2\rangle = \langle L_z^2\rangle$ from Eq. (10) when $m = n = 0$. Finally, using Eq. (12),

$$\langle L_{z_{mn}}^2\rangle = (2m+1)(2n+1)\frac{\hbar^2}{2}\left(\frac{1 + \epsilon_{0_{\text{ext}}}^2}{1 - \epsilon_{0_{\text{ext}}}^2}\right) - \frac{\hbar^2}{2}\,. \tag{15}$$

Hence the first non-zero moment, i.e., the second moment of OAM of higher-order Hermite–Gaussian modes is conserved and depends on the beam's extremal ellipticity, which is therefore also conserved.

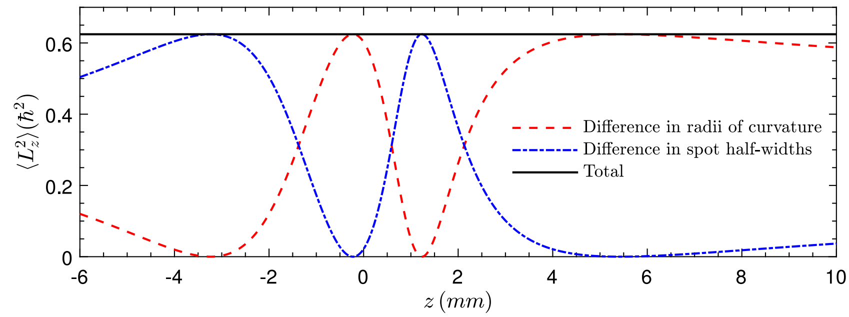

Fig. 3. Contributions to $\langle L_z^2 \rangle$ from the z-dependent difference in radii of curvature (red dashed curve) and in spot half-widths (blue dash-dotted curve) for the simple astigmatic Gaussian beam in the fundamental mode with the parameters described in Fig. 1. The sum of the two contributions (black solid line) is shown to be a constant upon propagation and is therefore conserved.

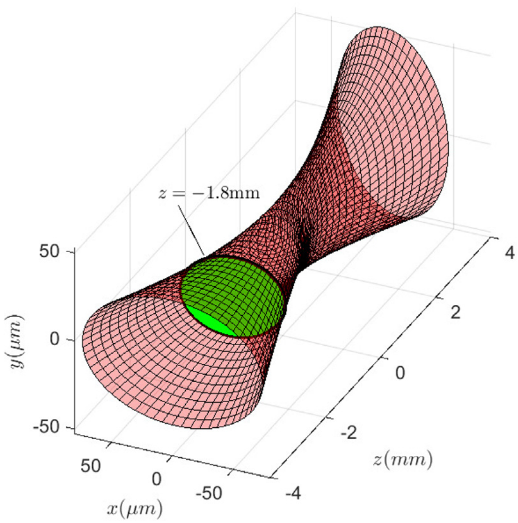

Fig. 4. Three dimensional beam width evolution of a general astigmatic beam with $2s = 2$ mm, $w_{0x} = 10\sqrt{2}\,\mu$m $= 14.14\,\mu$m, $w_{0y} = 15\sqrt{2}\,\mu$m $= 21.21\,\mu$m, $\lambda = 632.8$ nm and $\alpha = 0.1$ rad. The boundary of the beam is where the intensity drops to $1/e^2$ of its maximum value on the propagation axis. The cross-section of the beam at $z = -1.8$ mm, which is investigated next in Fig. 5, is highlighted in green.

The experimental technique for finding the spot half-widths for a Hermite–Gaussian beam has been discussed in [33].

5. General astigmatic Gaussian beam: fundamental mode

When the simple astigmatic Gaussian beam as described by Eq. (2) is rotated by a z-independent angle ϕ about the z-axis, it remains a valid solution of the paraxial wave equation (1). When ϕ is real, this only signifies a real rotation of the coordinate system, where the field pattern remains unaffected. However, when the angle of rotation is imaginary, i.e., $\phi = i\alpha$, where α is a constant of propagation, the beam profile changes. This new type of beam is defined as a general astigmatic beam, which can be generated from a simple astigmatic beam by rotating its wavefront relative to its beam spot along the xy plane so that they are no longer aligned.

5.1. Mode function

We will treat α as a basic, z-independent parameter to present the calculations for general astigmatic beams. The mode function of a general astigmatic beam can be written as [16]

$$u(x,y,z) = \sqrt{\frac{2}{\pi w_\xi w_\eta}} \exp\left[-\frac{\xi_w^2}{w_\xi^2} - \frac{\eta_w^2}{w_\eta^2} - ik\frac{\xi_R^2}{2R_\xi} - ik\frac{\eta_R^2}{2R_\eta} + i\left(\frac{\psi_x+\psi_y}{2}\right)\right], \tag{16}$$

where (ξ_w, η_w) and (ξ_R, η_R) are coordinate systems rotated at angles ϕ_w and ϕ_R, respectively, from the original (x,y) coordinate system (see Fig. 5). The angles ϕ_w and ϕ_R will be shown to be z-dependent and different from each other, indicating that the ellipses of constant intensity and the ellipses of constant phase change their orientation along the z-axis (see Fig. 6) instead of remaining constant and aligned like in the case of the simple astigmatic beam discussed in previous sections of this paper. The beam parameters w_ξ, w_η are the spot half-widths and R_ξ, R_η are the radii of curvature of the wavefront given by [16]

$$\frac{1}{w_{\xi,\eta}^2} = \frac{k}{4}\left(\omega_x + \omega_y \pm \sqrt{(\omega_x-\omega_y)^2\cosh^2 2\alpha + (\rho_x-\rho_y)^2\sinh^2 2\alpha}\right),$$
$$\frac{1}{R_{\xi,\eta}} = \frac{1}{2}\left(\rho_x + \rho_y \pm \sqrt{(\rho_x-\rho_y)^2\cosh^2 2\alpha + (\omega_x-\omega_y)^2\sinh^2 2\alpha}\right), \tag{17}$$

respectively. In the literature, for a general astigmatic Gaussian beam, $\rho_{x,y} = R_{x,y}^{-1} = (z \mp s)/[(z \mp s)^2 + z_{R_x,R_y}^2]$ and $\omega_{x,y} = (kw_{x,y}^2/2)^{-1} = (z_{R_x,R_y})/[(z \mp s)^2 + z_{R_x,R_y}^2]$ are conventionally used to explicitly express the real and imaginary parts, respectively, of the q parameters, i.e., $q_{x,y}^{-1} = \rho_{x,y} - i\omega_{x,y}$, and we will follow that convention. Using these notations, the angles of orientation of the ellipses of constant intensity and constant phase can be written as [16]

$$\tan 2\phi_w = \frac{\rho_x - \rho_y}{\omega_x - \omega_y}\tanh 2\alpha,$$
$$\tan 2\phi_R = -\frac{\omega_x - \omega_y}{\rho_x - \rho_y}\tanh 2\alpha. \tag{18}$$

and the difference $\Delta\phi = \phi_w - \phi_R$ is given by [16]

$$\tan 2\Delta\phi = -\sinh 2\alpha\,\cosh 2\alpha\left(\frac{\rho_x-\rho_y}{\omega_x-\omega_y} + \frac{\omega_x-\omega_y}{\rho_x-\rho_y}\right). \tag{19}$$

5.2. Ellipticity: general astigmatic beam

Following the simple astigmatic case [Eq. (5)] we can write the ellipticity of each rotating ellipse of constant intensity as [24]

$$\epsilon_\alpha(z) = \frac{w_\xi^2 - w_\eta^2}{w_\xi^2 + w_\eta^2}. \tag{20}$$

Using Eq. (17), we find that extremal ellipticity occurs at $z = z_{\text{ext}}$ [Eq. (7)] of the corresponding simple astigmatic Gaussian beam sharing

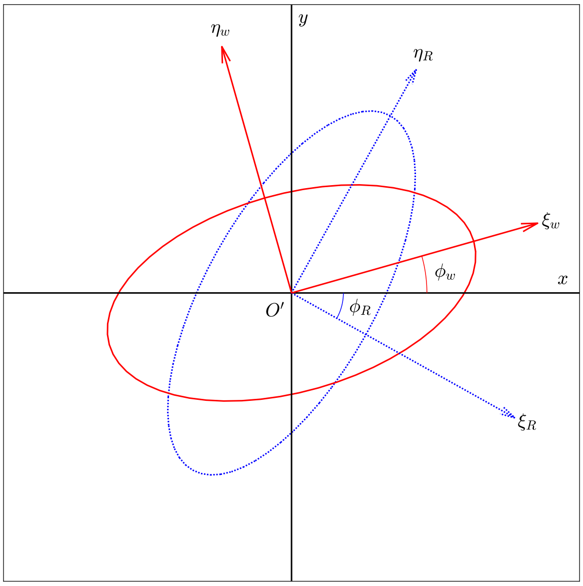

Fig. 5. Example of an ellipse of constant intensity (red solid) and an ellipse of constant phase (blue dotted) demonstrating their orientation with their respective coordinate systems (ξ_w, η_w) and (ξ_R, η_R) for the cross-section at $z = -1.8$ mm of the general astigmatic Gaussian beam shown in Fig. 4 [16]. In the figure, x-y is the original coordinate system and the angle of orientation ϕ_w of the ellipses of constant intensity is taking a positive value while ϕ_R of the ellipses of constant phase is taking a negative value.

the same initial parameters s, z_{R_x}, z_{R_y} and takes the form

$$\epsilon_{\text{ext}} = \epsilon_{0_{\text{ext}}} \cosh 2\alpha \ . \tag{21}$$

The constraints on α can be found in Appendix C.

The expectation value for orbital angular momentum of a general astigmatic beam in the fundamental mode can now be found from the integral

$$\langle L_{z_\alpha} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,y,z)^* L_{z_\alpha} u(x,y,z)\, d\xi_w\, d\eta_w \ , \tag{22}$$

where $L_{z_\alpha} = -i\hbar(\xi_w \partial/\partial\eta_w - \eta_w \partial/\partial\xi_w)$. We find,

$$\langle L_{z_\alpha} \rangle = \frac{\hbar k}{8}(w_\xi^2 - w_\eta^2)\left(\frac{1}{R_\xi} - \frac{1}{R_\eta}\right) \sin 2\Delta\phi \ . \tag{23}$$

This result has been confirmed previously [34–36]. The parameters in Eq. (23) are still z-dependent. Applying Eq. (17) and (19), however, demonstrates that the mean OAM is indeed constant throughout propagation,

$$\langle L_{z_\alpha} \rangle = \hbar \frac{\sinh 2\alpha \ \cosh 2\alpha \ (A^2 + 4\Omega^2)}{4 - 4\Omega^2 \cosh^2 \alpha - A^2 \sinh^2 2\alpha} \ , \tag{24}$$

Finally, by using Eq. (21), we can write, simply,

$$\langle L_{z_\alpha} \rangle = \hbar \left(\frac{\epsilon_{\text{ext}}^2}{1 - \epsilon_{\text{ext}}^2} \right) \tanh 2\alpha \ . \tag{25}$$

Higher moments of OAM, also sole functions of the extremal ellipticity, are presented in Appendix B.2.

6. General astigmatic Gaussian beam: higher order modes

In this section, we will extend the general astigmatic Gaussian beam to its higher-order Hermite–Gaussian modes, whose mode function at the cross-section where extremal ellipticity occurs is given by [see Appendix D]

$$u_{mn}(x,y,z_{\text{ext}}) = \sqrt{\frac{1}{\pi w_\xi(z_{\text{ext}}) w_\eta(z_{\text{ext}}) 2^{m+n-1} m! n!}}\, H_m\left(\frac{\sqrt{2}x}{w_\xi(z_{\text{ext}})}\right) H_n\left(\frac{\sqrt{2}y}{w_\eta(z_{\text{ext}})}\right)$$
$$\times \exp\left[-\frac{x^2}{w_\xi^2(z_{\text{ext}})} - \frac{y^2}{w_\eta^2(z_{\text{ext}})} - ik\frac{(x-y)^2}{4R_\xi(z_{\text{ext}})} - ik\frac{(x+y)^2}{4R_\eta(z_{\text{ext}})} \right] . \tag{26}$$

For this mode function, we find

$$\langle L_{z_{\alpha_{mn}}} \rangle = (n+m+1)\langle L_{z_{\alpha_{00}}} \rangle + (m-n)\langle L_{z_{\alpha_{00}}} \rangle \frac{w_\xi^2(z_{\text{ext}}) + w_\eta^2(z_{\text{ext}})}{w_\xi^2(z_{\text{ext}}) - w_\eta^2(z_{\text{ext}})} \ , \tag{27}$$

where $\langle L_{z_{\alpha_{00}}} \rangle = \langle L_{z_\alpha} \rangle$ is the known OAM of a general astigmatic beam in the fundamental mode [Eq. (24)]. Using Eq. (25) now gives

$$\langle L_{z_{\alpha_{mn}}} \rangle = \hbar \left[\frac{(n+m+1)\epsilon_{\text{ext}}^2 + (m-n)\epsilon_{\text{ext}}}{1 - \epsilon_{\text{ext}}^2} \right] \tanh 2\alpha \ . \tag{28}$$

Therefore, we have directly related the extremal ellipticity of the fundamental as well as higher-order modes of a general astigmatic Gaussian beam to the first moment of OAM, which is conserved upon propagation through a stigmatic lens.

7. Conclusion

Our paper provides a fundamental explanation for the observed constancy of the extremal ellipticity [24] by relating it to the conservation of the OAM distribution of the beam, allaying concerns that

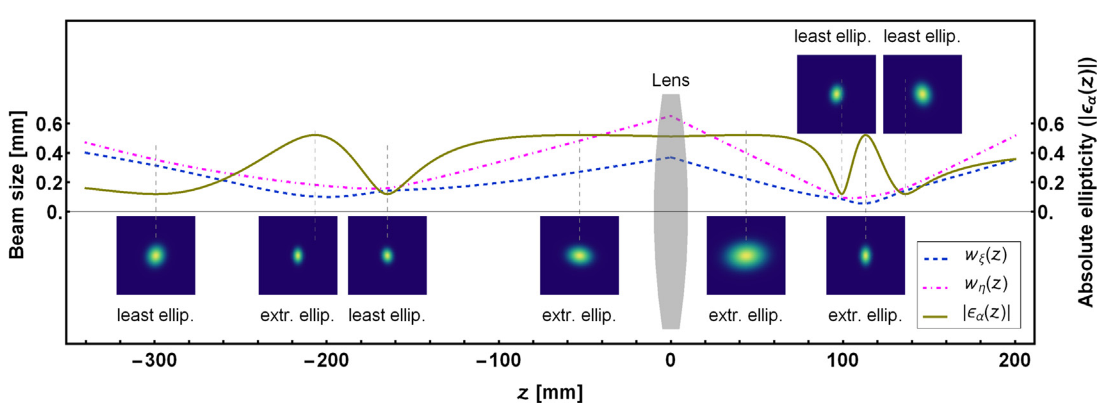

Fig. 6. The evolution of the intensity profile for a general astigmatic Gaussian beam with the same parameters as the simple astigmatic beam (detailed in Fig. 2) and at the imaginary rotation angle $\alpha = 0.1$ rad propagating through a symmetric lens placed at $z = 0$ mm with focal length $f = 250$ mm. The beam-width variation [both w_ξ (dashed blue) and w_η (dot-dashed magenta)] and the resultant ellipticity modulation (solid olive) are shown along the propagation axis. Since the ξ_w- and η_w-axes are always rotating, it is no longer meaningful to keep track of the sign of ellipticity, hence its absolute value ($|\ \epsilon_\alpha(z)\ |$) is plotted instead. The beam profile is aligned with the x- and y-axes at extremal ellipticity (extr. ellip.), but is generally rotated at an angle elsewhere, such as at positions of least ellipticity (least ellip.), i.e., smallest absolute ellipticity, as displayed in the figure.

this is accidental or only applicable under very limited circumstances. On the one hand this represents a practical advance in beam shaping and control because while rarely expounded in the scientific literature, beam quality and beam roundness in particular, are of great importance in many precision industrial laser processes and substantial optical design efforts are undertaken to ensure sufficiently well controlled optical performance. On the other hand, connecting the extremal ellipticity to as fundamental a concept as orbital angular momentum and demonstrating the effective equivalence of the two not only for single astigmatic Gaussian beams, but general astigmatic, and even general astigmatic Hermite–Gaussian beams raises the question if and how this insight may be extendable to even more general cases, be it the inclusion of higher order aberrations, partially coherent Gauss–Schell model beams or even incoherent superpositions of several transverse modes. Finally, control of extremal ellipticity allows us to shape the beam OAM distribution itself. We expect our results to be relevant to fields where the measurement and management of beam ellipticities play a crucial role such as laser micromachining, imaging, optical trapping, and inertial confinement fusion.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Simple astigmatic coherent Gaussian beam

The beam parameters in the mode function of simple astigmatic Gaussian beam [Eq. (4)] have the following forms [28,29]

$$\begin{aligned} w_{x,y} &= w_{0x,0y}\sqrt{1+\frac{(z\mp s)^2}{z^2_{R_x,R_y}}}\,, \\ R_{x,y} &= z\mp s+\frac{z^2_{R_x,R_y}}{z\mp s}\,, \\ \psi_{0x,0y} &= \tan^{-1}\left(\frac{z\mp s}{z_{R_x,R_y}}\right)\,, \end{aligned} \tag{A.1}$$

The astigmatism A is defined as

$$A=\frac{2s}{\frac{1}{2}(z_{R_x}+z_{R_y})}=\frac{4s}{z_{R_x}+z_{R_y}}\,. \tag{A.2}$$

Without loss of generality, we will choose s to be positive and A thus varies in the range $0<A<\infty$. Following [24], the Asymmetry Ω takes the form

$$\Omega=\frac{w^2_{0x}-w^2_{0y}}{w^2_{0x}+w^2_{0y}}=\frac{z_{R_x}-z_{R_y}}{z_{R_x}+z_{R_y}}\,. \tag{A.3}$$

It is z-independent and varies as $-1<\Omega<+1$.

Appendix B. Astigmatic Gaussian beams — higher order moments

The pth order moment of OAM can be found by evaluating the integral

$$\langle L^p_z\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^*(x,y,z)L^p_z u(x,y,z)\,d\xi\,d\eta\,, \tag{B.1}$$

B.1. OAM moments: Simple astigmatic beam in the fundamental mode

The expectation value of any odd-order moments of OAM of a simple astigmatic beam will evaluate to zero. But for even orders, the expectation value of the OAM-moment will be non-zero. For example,

$$\langle L^4_z\rangle=4\hbar^2\langle L^2_z\rangle+9\langle L^2_z\rangle^2\,. \tag{B.2}$$

By using Eq. (12), $\langle L^4_z\rangle$ can be directly related to the extremal ellipticity $\epsilon_{0_{\rm ext}}$.

B.2. OAM moments: General astigmatic beam in the fundamental mode

The second moment is given by

$$\langle L^2_{z_\alpha}\rangle=\frac{2\hbar\langle L_{z_\alpha}\rangle}{\tanh(4\alpha)}+3\langle L_{z_\alpha}\rangle^2\,, \tag{B.3}$$

where $\langle L_{z_\alpha}\rangle$ is given in Eq. (23). Using Eq. (25), we get

$$\langle L^2_{z_\alpha}\rangle=\hbar^2(1+\tanh^2 2\alpha)\left(\frac{\epsilon^2_{\rm ext}}{1-\epsilon^2_{\rm ext}}\right)+3\hbar^2\tanh^2 2\alpha\frac{\epsilon^4_{\rm ext}}{\left(1-\epsilon^2_{\rm ext}\right)^2}\,. \tag{B.4}$$

The third moment is given by

$$\langle L^3_{z_\alpha}\rangle=-2\hbar^2\langle L_{z_\alpha}\rangle+\frac{9\hbar(1+\tanh^2 2\alpha)\langle L_{z_\alpha}\rangle^2}{\tanh 2\alpha}+15\langle L_{z_\alpha}\rangle^3, \tag{B.5}$$

and by relating to Eq. (25), can be expressed in terms of $\epsilon_{\rm ext}$ and α.

Appendix C. Relationship of α and extremal ellipticity of general astigmatic beam

Arnaud et al. [16] had found that the range of allowed α values such that the mode function in Eq. (16) can be normalized is given by

$$\cosh^2(2\alpha)\le\frac{4s^2+(z_{R_x}+z_{R_y})^2}{4s^2+(z_{R_x}-z_{R_y})^2}. \tag{C.1}$$

We have found that this condition can be written very compactly as

$$\cosh^2 2\alpha\le\frac{1}{\epsilon^2_{0_{\rm ext}}}=\frac{A^2+4}{A^2+4\Omega^2}\,. \tag{C.2}$$

A consequence of Eq. (C.2) is that for a given $\epsilon_{0_{\rm ext}}$, there is a maximum allowed value of α to maintain this inequality.

Appendix D. Mode function of higher-order general astigmatic Gaussian beam

We choose to generate higher-order Hermite–Gaussian modes for a general astigmatic beam at a convenient location on the z axis [34], namely the positive sign solution of $z=z_{\rm ext}$ from Eq. (7) for sake of simplicity since $\phi_w=0$ and $\phi_R=\pi/4$ (see Section 5.2). The mode function for the fundamental mode of a general astigmatic beam at $z_{\rm ext}$ is written as

$$\begin{aligned} u(x,y,z_{\rm ext})=\sqrt{\frac{2}{\pi w_\xi(z_{\rm ext})w_\eta(z_{\rm ext})}}\exp\Bigg[&-\frac{x^2}{w^2_\xi(z_{\rm ext})}-\frac{y^2}{w^2_\eta(z_{\rm ext})} \\ &-ik\frac{(x-y)^2}{4R_\xi(z_{\rm ext})}-ik\frac{(x+y)^2}{4R_\eta(z_{\rm ext})}\Bigg]\,. \end{aligned} \tag{D.1}$$

Here, we have suppressed the Gouy phase since it is purely a function of z while we are concerned with the transformation of the (x,y)-dependent parts. Eq. (D.1) can be expressed more compactly using the beam parameter matrix $Q(z)$ as [34]

$$u(x,y,z_{\rm ext})=\sqrt{\frac{2}{\pi w_\xi(z_{\rm ext})w_\eta(z_{\rm ext})}}\exp\left[-\frac{1}{2}R^\dagger Q(z_{\rm ext})R\right]\,, \tag{D.2}$$

where $Q(z_{\rm ext})=Q_0(z_{\rm ext})+iQ_1(z_{\rm ext})$ and

$$Q_0(z_{\rm ext})=2\begin{bmatrix}1/w^2_\xi(z_{\rm ext}) & 0\\ 0 & 1/w^2_\eta(z_{\rm ext})\end{bmatrix}\,, \tag{D.3}$$

$$Q_1(z_{\text{ext}}) = \frac{k}{2}\begin{bmatrix} 1/R_\xi(z_{\text{ext}}) + 1/R_\eta(z_{\text{ext}}) & 1/R_\eta(z_{\text{ext}}) - 1/R_\xi(z_{\text{ext}}) \\ 1/R_\eta(z_{\text{ext}}) - 1/R_\xi(z_{\text{ext}}) & 1/R_\xi(z_{\text{ext}}) + 1/R_\eta(z_{\text{ext}}) \end{bmatrix} . \quad \text{(D.4)}$$

Defining the raising operators at any position z on the propagation axis as [34]

$$A^\dagger(z) = \frac{1}{\sqrt{2}}\beta^*(z)(Q^*(z)R + iP), \quad \text{(D.5)}$$

where $A^\dagger(z) = [a_x^\dagger(z)\ a_y^\dagger(z)]^T$, $R = [x\ y]^T$, and $P = [P_x\ P_y]^T$ are the vectors of the raising operators, position, and momentum, respectively, and $\beta(z)$ is a 2×2 matrix determining which higher-order modes are generated. It has to satisfy $\beta^\dagger(z)\beta(z) = Q_0^{-1}(z)$. At $z = z_{\text{ext}}$, we choose

$$\beta(z_{\text{ext}}) = \frac{1}{\sqrt{2}}\begin{bmatrix} w_\xi(z_{\text{ext}}) & 0 \\ 0 & w_\eta(z_{\text{ext}}) \end{bmatrix} . \quad \text{(D.6)}$$

to generate Hermite–Gaussian modes along the x and y axes. By evaluating Eq. (D.5) at $z = z_{\text{ext}}$, we obtain the raising operators

$$\begin{aligned} a_x^\dagger(z_{\text{ext}}) &= \frac{w_\xi(z_{\text{ext}})}{2}\left[x\left(\frac{2}{w_\xi^2(z_{\text{ext}})} - \frac{ik}{2R_\xi(z_{\text{ext}})} - \frac{ik}{2R_\eta(z_{\text{ext}})}\right) - \frac{\partial}{\partial x} \right. \\ &\qquad \left. - \frac{iky}{2}\left(\frac{1}{R_\eta(z_{\text{ext}})} - \frac{1}{R_\xi(z_{\text{ext}})}\right)\right] , \\ a_y^\dagger(z_{\text{ext}}) &= \frac{w_\eta(z_{\text{ext}})}{2}\left[y\left(\frac{2}{w_\eta^2(z_{\text{ext}})} - \frac{ik}{2R_\xi(z_{\text{ext}})} - \frac{ik}{2R_\eta(z_{\text{ext}})}\right) - \frac{\partial}{\partial y} \right. \\ &\qquad \left. - \frac{ikx}{2}\left(\frac{1}{R_\eta(z_{\text{ext}})} - \frac{1}{R_\xi(z_{\text{ext}})}\right)\right] . \end{aligned} \quad \text{(D.7)}$$

The (m,n) higher-order Hermite–Gaussian modes of a general astigmatic beam aligned with the (x,y) coordinate system can then be generated from a general astigmatic beam in the fundamental mode at $z = z_{\text{ext}}$ by applying the raising operators as [34]

$$u_{m,n}(x,y,z_{\text{ext}}) = [a_x^\dagger(z_{\text{ext}})]^m [a_y^\dagger(z_{\text{ext}})]^n u(x,y,z_{\text{ext}}) . \quad \text{(D.8)}$$

This procedure yields the mode function of Eq. (26).

References

[1] L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, J.P. Woerdman, Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes, Phys. Rev. A 45 (11) (1992) 8185–8189.

[2] A.M. Yao, M.J. Padgett, Orbital angular momentum: origins, behavior and applications, Adv. Opt. Photon. 3 (2) (2011) 161–204.

[3] H. He, M.E.J. Friese, N.R. Heckenberg, H. Rubinsztein-Dunlop, Direct observation of transfer of angular momentum to absorptive particles from a laser beam with a phase singularity, Phys. Rev. Lett. 75 (5) (1995) 826–829.

[4] J. Courtial, K. Dholakia, D.A. Robertson, L. Allen, M.J. Padgett, Measurement of the rotational frequency shift imparted to a rotating light beam possessing orbital angular momentum, Phys. Rev. Lett. 80 (15) (1998) 3217–3219.

[5] J. Courtial, D.A. Robertson, K. Dholakia, L. Allen, M.J. Padgett, Rotational frequency shift of a light beam, Phys. Rev. Lett. 81 (22) (1998) 4828–4830.

[6] D.G. Grier, A revolution in optical manipulation, Nature 424 (6950) (2003) 810–816.

[7] T. Kuga, Y. Torii, N. Shiokawa, T. Hirano, Y. Shimizu, H. Sasada, Novel optical trap of atoms with a doughnut beam, Phys. Rev. Lett. 78 (25) (1997) 4713–4716.

[8] S. Bernet, A. Jesacher, S. Fürhapter, C. Maurer, M. Ritsch-Marte, Quantitative imaging of complex samples by spiral phase contrast microscopy, Opt. Express 14 (9) (2006) 3792–3805.

[9] K.I. Willig, S.O. Rizzoli, V. Westphal, R. Jahn, S.W. Hell, STED microscopy reveals that synaptotagmin remains clustered after synaptic vesicle exocytosis, Nature 440 (7086) (2006) 935–939.

[10] A. Mair, A. Vaziri, G. Weihs, A. Zeilinger, Entanglement of the orbital angular momentum states of photons, Nature 412 (6844) (2001) 313–316.

[11] J. Wang, J.-Y. Yang, I.M. Fazal, N. Ahmed, Y. Yan, H. Huang, Y. Ren, Y. Yue, S. Dolinar, M. Tur, A.E. Willner, Terabit free-space data transmission employing orbital angular momentum multiplexing, Nature Photon 6 (7) (2012) 488–496.

[12] G. Vicidomini, P. Bianchini, A. Diaspro, STED super-resolved microscopy, Nature Methods 15 (3) (2018) 173–182.

[13] M. Padgett, R. Bowman, Tweezers with a twist, Nature Photon 5 (6) (2011) 343–348.

[14] M. Gecevičius, R. Drevinskas, M. Beresna, P.G. Kazansky, Single beam optical vortex tweezers with tunable orbital angular momentum, Appl. Phys. Lett. 104 (23) (2014) 231110.

[15] J. Courtial, K. Dholakia, L. Allen, M.J. Padgett, Gaussian beams with very high orbital angular momentum, Opt. Commun. 144 (4–6) (1997) 210–213.

[16] J.A. Arnaud, H. Kogelnik, Gaussian light beams with general astigmatism, Appl. Opt. 8 (8) (1969) 1687–1693.

[17] J. Serna, P.M. Mejías, R. Martínez-Herrero, Rotation of partially coherent beams propagating through free space, Opt. Quant. Electron 24 (9) (1992) S873–S880.

[18] J. Serna, J.M. Movilla, Orbital angular momentum of partially coherent beams, Opt. Lett. 26 (7) (2001) 405–407.

[19] G. Nemes, Spatial laser beam characterization and propagation through ABCD-type optical systems using the second-order moments method, in: S. Martellucci, M. Santarsiero (Eds.), Free and Guided Optical Beams, World Scientific, Singapore, 2004, pp. 61–76.

[20] V.V. Kotlyar, A.A. Kovalev, A.P. Porfirev, Vortex Hermite–Gaussian laser beams, Opt. Lett. 40 (5) (2015) 701–704.

[21] V.V. Kotlyar, A.A. Kovalev, A.P. Porfirev, E.S. Kozlova, Three different types of astigmatic Hermite-Gaussian beams with orbital angular momentum, J. Opt. 21 (11) (2019) 115601.

[22] E. Abramochkin, V. Volostnikov, Beam transformations and nontransformed beams, Opt. Commun. 83 (1–2) (1991) 123–135.

[23] A.E. Siegman, How to (maybe) measure laser beam quality, in: M.W. Dowley (Ed.), Diode Pumped Solid State Lasers: Applications and Issues, OSA, Washington, DC, 1998, MQ1.

[24] H.W. Lo, J. Kleinert, M. Kleinert, Conservation of extremal ellipticity and analytical expressions for astigmatism and asymmetry in Gaussian beam transformations, Appl. Opt. 56 (9) (2017) 2523–2528.

[25] Y. Sun, B.E. Nilsen, D.M. Hemenway, L. Sun, Laser pulse picking employing controlled AOM loading, US Patent 6947454B2, September 20, 2005.

[26] A.T. Kitai, A. Mank, P.A. Labelle, A. Naumov, I.J. Miller, R.G. Parker, Laser trim motion, calibration, imaging, and fixturing techniques, US Patent 2005/0205778 A1, September 22, 2005.

[27] A.E. Siegman, Lasers, University Science Books, Sausalito, CA, 1986.

[28] H.A. Haus, Waves and Fields in Optoelectronics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1984.

[29] J. Peatross, M. Ware, Physics of Light and Optics, Brigham Young University, 2015.

[30] S.J. Van Enk, G. Nienhuis, Commutation rules and eigenvalues of spin and orbital angular momentum of radiation fields, J. Modern Opt. 41 (5) (1994) 963–977.

[31] G. Nienhuis, L. Allen, Paraxial wave optics and harmonic oscillator, Phys. Rev. A 48 (1) (1993) 656–665.

[32] V.V. Kotlyar, A.A. Kovalev, A.P. Porfirev, Asymmetric Gaussian optical vortex, Opt. Lett. 42 (1) (2017) 139–142.

[33] O. Mata-Mendez, Two methods to determine the Hermite-Gaussian beam radius by means of aperiodic rulings, Rev. Mexicana Fís. 56 (2) (2010) 141–146.

[34] J. Visser, G. Nienhuis, Orbital angular momentum of general astigmatic modes, Phys. Rev. A 70 (1) (2004) 013809.

[35] C. Gao, G. Wei, H. Weber, Orbital angular momentum of the laser beam and the second order intensity moments, Sci. China A 43 (12) (2000) 1306–1311, We detected a sign discrepancy in Eq. (33) of the cited paper: there should be a plus sign instead of a minus sign before $\omega_x^2\omega_y^2$ inside the square bracket. With the sign adjustment, this equation agrees with Eq. (23) of our paper. The mistake might just be a typographical error, since Eq. (32) before Eq. (33) in Gao, Wei and Weber's paper agrees with Eq. (23) of our paper.

[36] V.V. Kotlyar, A.A. Kovalev, A.P. Porfirev, Astigmatic laser beams with a large orbital angular momentum, Opt. Express 26 (1) (2018) 141–156.